# Propagation of decoherence in distributed quantum systems

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We study the decohering influence of measurement performed locally on some region of a distributed quantum system. We demonstrate that the local decohering perturbation exerted on the measured region can propagate over the system in the form of decoherence wave. This process refers to the local properties at different points inside the system, and is fundamentally different from such processes as the wave function collapse, which concern the properties of the system as a whole. As an example, we consider the gas of bosons forming Bose-Einstein condensate. We show that the decoherence propagation in ideal Bose-gas is a diffusion process, while in the gas of weakly interacting bosons it is a wave traveling with the sound velocity. The results obtained can be checked in real experiments using, e.g. the Bose-gas of supercooled trapped ions.

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Theory of quantum measurement founded [1] as early as in the end of 1920ths still remains in the focus of interest of physicists (see, e.g. [2,3] and references therein). The act of measurement leads to a "reduction of a wave function" of a quantum system, which is an instant event embracing the whole system at once and propagating through the system with infinite velocity. This impressive quantum effect, widely known as Einstein-Podolsky-Rosen paradox, demonstrate peculiar properties of quantum correlations which have been a subject of much controversy [4], and have lead to a deep revision of phylosophical foundations of physics.

Unitary evolution of a system prepared initially in a pure quantum state is interrupted by the measurement. The system experiences sudden decohering action of the measuring apparatus, the density matrix describing an ensemble of such systems changes radically (it suddenly ceases to be a projection operator) and entropy rises instantly. These effects are very well known, they are characteristic for any kind of quantum system. What is most important, they describe the changes experienced by the system as a whole, i.e. the changes of global properties of quantum systems. However, for a *distributed* quantum system a natural question arises: what happens with a distributed system subjected to a *local* measurement? How *internal* properties of one region of the system are affected by a measurement performed on some other region? Indeed, local measurement is a real physical action exerted on some region of the system. Due to this action, the measured region of the system changes its physical state and appears to possess other physical properties than the rest of the system. Further evolution of the measured region, in general, affects other regions, e.g. due to interactions present in the system, so that the decohering influence of the local measurement can propagate through the system in the form of *decoherence wave*. Therefore, apart from the wave function collapse, which is not caused by any interactions in the system and is an instant event, there are real physical effects of propagation of the perturbation induced by measurement. These effects constitute the subject of the present paper.

Along with being interesting from the fundamental point of view, the phenomenon of decoherence propagation can be also of crucial importance for design of quantum computers. Such a computer is a system of interacting quantum entities, representing quantum bits (qubits). Fault-tolerant procedures of quantum computations involve measurements performed on some qubits and it is important to know how it may affect the states of other qubits [5]. Moreover, decoherence is introduced by a dissipative environment of qubits, so that analysis of decoherence propagation can provide pathways to minimize unwanted impacts on performance of the computer.

A good example of a distributed system in a pure quantum state is Bose-Einstein condensate of an ideal or weakly non-ideal gas of bosons. Such a system can be implemented experimentally as a gas of trapped atoms cooled down to very low temperatures [6]. Suppose we measure the number of particles in some region of space, e.g. by focusing X-ray beam in this region and studying the scattering cross-section. If two measurements are done *simultaneously* at two different parts of the trap we obtain the trivial result corresponding to the ground-state wavefunction of the condensate. But if the second measurement is carried out after some delay then its results change, visualizing the propagation of the perturbation introduced by the first measurement. In such form, the effects considered here can be, in principle, investigated in real physical experiments.

To study quantitatively the effect of decoherence propagation, let us consider first an ideal Bose-gas confined by external fields and described by the Hamiltonian $H$. Let us denote the one-paricle wavefunctions as $\varphi_\mu(\mathbf{r})$ and corresponding one-particle energies as $E_\mu$, where $\mu = 0$ stands for the ground state having minimal energy $E_0 = 0$. Introducing the boson creation and annihilation operators $\alpha_\mu^\dagger$ and $\alpha_\mu$, the ground-state eigenfunction of the system of $M$ bosons can be written as

$$|\Psi\rangle = \frac{1}{\sqrt{M!}}\left(\alpha_0^\dagger\right)^M |0\rangle, \tag{1}$$

where $|0\rangle$ is the vacuum state. For simplicity, we can consider the ion trap as being divided into a large number $N_c$ of small cells each having the volume $V_0$ (it can be considered as the volume of the region where X-ray beam is focused), satisfying the relation $V_0 \ll V$, where $V$ is the total volume of the trap. Then, the coordinate $\mathbf{r}$ is understood as a discrete quantity (the number of a cell). This is similar to a general practice of solid-state theory, where $V_0$ is analogous to the volume of elementary cell of the crystal [7]. Note that in so doing, we restrict the number of one-particle states, so that this number is equal to $N_c$, which is finite, though very large. This corresponds to the fact that the number of states inside the first Brillouin zone equals to the number of lattice cells.

At the instant $t = 0$ we perform measurement of the number of bosons in the cell $\mathbf{r} = 0$. This observable is represented by the operator $N = a^\dagger(0)a(0)$, where

$$a(\mathbf{r}) = \sum_\mu \varphi_\mu(\mathbf{r})\alpha_\mu. \tag{2}$$

is the boson field operator. Eigenvalues of the operator $N$ are $n = 0, 1, 2...$ and, suppose, the measurement has given us one of them. It corresponds to the action of the operator $W_n$ on the system, where

$$W_n = \delta_{n,N} = \int_0^{2\pi} \frac{d\phi}{2\pi} \exp\left[i\phi(n - N)\right]. \tag{3}$$

Further development of the system is to be described by density matrix of the system $U(t)$, since the measurement interrupts unitary evolution and casts the system into mixed quantum state. According to the standard theory of measurements [1,3], the density matrix at the time $t$ is

$$U(t) = \sum_{n=0}^{\infty} \exp(-iHt) W_n U_{\rm in} W_n^\dagger \exp(iHt), \tag{4}$$

where $U_{\rm in} = |\Psi\rangle\langle\Psi|$ is the density matrix before the measurement.

To trace propagation of decoherence in the system, we study evolution of the one-particle density matrix

$$\rho(\mathbf{r}, \mathbf{r}', t) = {\rm Tr}\left[U(t)a^\dagger(\mathbf{r}')a(\mathbf{r})\right]. \tag{5}$$

This quantity is a very suitable tool for our investigation describing local properties of Bose-Einstein condensate. In particular, average number of particles resulted from second measurement, which is performed at the point $\mathbf{r}$ at the instant $t$, is given by the value $\rho(\mathbf{r}, \mathbf{r}, t)$.

To simplify calculations, we use the fact that the total number of particles is large, $M \gg 1$, so that operators $\alpha_0$ and $\alpha_0^\dagger$ acting on the state $|\Psi\rangle$ can be replaced by the number $\sqrt{M}$ with relative accuracy $1/\sqrt{M}$; this is a standard approximation in the theory of Bose-Einstein condensation [9]. Therefore, Eq. (2) can be rewritten as

$$a(\mathbf{r}) = \sqrt{n_B(\mathbf{r})} + \bar{a}(\mathbf{r}), \qquad \bar{a}(\mathbf{r}) = \sum_{\mu\neq 0} \varphi_\mu(\mathbf{r})\alpha_\mu \tag{6}$$

where $n_B(\mathbf{r}) = M\varphi_0^2(\mathbf{r})$ is the average number of condensate particles contained in the volume $V_0$ at the cell $\mathbf{r}$. The expression for the one-particle density matrix can be written as

$$\rho(\mathbf{r}, \mathbf{r}', t) = \sum_{n=0}^{\infty} \rho_n(\mathbf{r}, \mathbf{r}', t), \tag{7}$$
$$\rho_n(\mathbf{r}, \mathbf{r}', t) = \langle\Psi|W_n^\dagger a^\dagger(\mathbf{r}', t)a(\mathbf{r}, t)W_n|\Psi\rangle,$$

where $a(\mathbf{r}, t) = \exp(iHt)a(\mathbf{r})\exp(-iHt)$. Operator product in Eq. (7) is to be ordered normally, i.e. it is to be rewritten in such a way that all $a^\dagger$ stand to the left of all $a$ in each term of Taylor series expansion. In so doing, we take into account that

$$\left[a(\mathbf{r}, t), \bar{a}^\dagger(0)\right] = \sum_{\mu\neq 0} \varphi_\mu(\mathbf{r})\varphi_\mu^*(0){\rm e}^{-iE_\mu t} \equiv g(\mathbf{r},t). \tag{8}$$

Note that for a system containing a large number of particles $M \gg 1$, the function $g(\mathbf{r}, t)$ can be replaced by the Green's function

$$G(\mathbf{r}, t) = \sum_\mu \varphi_\mu(\mathbf{r})\varphi_\mu^*(0){\rm e}^{-iE_\mu t} \tag{9}$$

with accuracy of order of $1/M$, since $G(\mathbf{r}, t) = g(\mathbf{r}, t) + \varphi_0(\mathbf{r})\varphi_0^*(0)$. Having the calculations done, we obtain

$$\begin{aligned}\rho_n(\mathbf{r}, \mathbf{r}', t) = p_n &\left[\sqrt{n_B(\mathbf{r})} - G(\mathbf{r}, t)\sqrt{n_0}\right] \\ &\times \left[\sqrt{n_B(\mathbf{r}')} - G^*(\mathbf{r}', t)\sqrt{n_0}\right] \\ &+ p_{n-1} n_0 G(\mathbf{r}, t)G^*(\mathbf{r}', t),\end{aligned} \tag{10}$$

where $n_0 = n_B(0)$, and $p_n = {\rm e}^{-n_0}\, n_0^n/(n!)$ is Poisson distribution function. Summation over $n$ can be performed explicitely, yielding the answer

$$\begin{aligned}\rho(\mathbf{r}, \mathbf{r}', t) = &\sqrt{n_B(\mathbf{r})n_B(\mathbf{r}')} - G^*(\mathbf{r}', t)\sqrt{n_B(\mathbf{r})n_0} \\ &- G(\mathbf{r}, t)\sqrt{n_B(\mathbf{r}')n_0} + 2n_0 G^*(\mathbf{r}', t)G(\mathbf{r}, t).\end{aligned} \tag{11}$$

The result (11) for one-particle density matrix demonstrates that the measurement made at the point $\mathbf{r} = 0$ produces a decohering perturbation which propagates gradually over the whole trap in the form of decoherence wave. Temporal behavior of this propagation is governed by the Green's function $G(\mathbf{r}, t)$. For the gas consisting of free Bose-particles of mass $m$ [8]

$$G(\mathbf{r}, t) = V_0\left(\frac{m}{2\pi i\hbar t}\right)^{3/2} \exp\left(\frac{im\mathbf{r}^2}{2\pi\hbar t}\right) \tag{12}$$

at the distances $r \gg V_0^{1/3}$, i.e. the decoherence propagation is a diffusion process.

For understanding of the results, it is important to realize that the measurement constitutes a strong real physical action applied to the system, accompanied, for example, by a finite (and rather large) amount of energy $\Delta E$ transferred into the system. This quantity can be calculated:

$$\Delta E = 2n_0 \sum_{\mu \neq 0} E_\mu \left|\varphi_\mu(0)\right|^2, \tag{13}$$

where we have to take into account the fact that the number of wavefunctions $\varphi_\mu$ is finite and equal to the number of cells $N_c = V/V_0$ in the volume of the trap. Also, the entropy of the system is raised; being initially zero, the entropy after the measurement is

$$S = -\sum_{n=0}^{\infty} p_n \ln p_n, \tag{14}$$

what is physically very clear.

Therefore, the results obtained can be qualitatively interpreted as follows. The measurement performed at $\mathbf{r} = 0$ means that the measurement apparatus (whatever it is) localizes some number of particles within the cell $\mathbf{r} = 0$. The particles affected by the measurement acquire rather large momenta, of order of $\hbar/V_0^{1/3}$; the average number of such particles is $n_0 = n_B(0)$. Immediately after being localized, these particles start to propagate over the trap, and their propagation is governed by the Green's function (12), i.e. it constitutes a diffusion process. Of course, because of indistinguishability of particles in the trap, we can not say that these are "the same" particles which has been measured at $\mathbf{r} = 0$, so that the effect we consider is not a motion of some well-separated particles in the trap, but rather it is propagation of the decohering influence of the measurement through the system.

Another interesting feature of the decoherence propagation effect can be illustrated best by the gas of Bose-particles trapped in a parabolic external potential, so that each particle is represented by an isotropic harmonic oscillator of eigenfrequency $\Omega$. In this case, provided that $r \gg V_0^{1/3}$ and $V_0 \ll (\hbar/\Omega)^{3/2} \sim V$, the Green's function has the form [8]

$$G(\mathbf{r}, t) = V_0 \left(\frac{\Omega}{2\pi i\hbar \sin \Omega t}\right)^{3/2} \exp\left(\frac{i\Omega \mathbf{r}^2}{2\pi\hbar} \cot \Omega t\right) \tag{15}$$

where the particles are assumed to have unitary mass. This function is periodic in time with the temporal period $2\pi/\Omega$. Therefore, the decoherence propagation is also exactly periodic in time with the same period. In the general case of Bose-gas trapped in a finite volume and/or arbitrary potential, the decoherence propagation becomes a quasiperiodic process, according to Eq. (8). Moreover, the periodicity (or quasiperiodicity) is destroyed due to the decohering influence of dissipative environment.

Up to now, we have considered the system of noninteracting bosons. How the process of decoherence propagation changes if an interaction between particles is present? To investigate this problem, let us study the gas of weakly interacting bosons, (weakly non-ideal Bose-gas), contained in a trap of large volume $V$. We assume no external potential acting on the particles, so that the one-particle states are simple plane waves

$$\varphi_{\mathbf{k}}(\mathbf{r}) = \sqrt{\frac{V_0}{V}} \exp(i\mathbf{k}\mathbf{r}), \tag{16}$$

where the normalization takes into account the fact that the trap is divided into cells of volume $V_0 \ll V$. This system is described by the Hamiltonian

$$\begin{aligned} H = &\sum_{\mathbf{k}} E_{\mathbf{k}} \alpha_{\mathbf{k}}^\dagger \alpha_{\mathbf{k}} \\ &+ \frac{1}{2V} \sum_{\mathbf{k}_1+\mathbf{k}_2=\mathbf{k}_1'+\mathbf{k}_2'} v(\mathbf{k}_1 - \mathbf{k}_1') \alpha_{\mathbf{k}_1'}^\dagger \alpha_{\mathbf{k}_2'}^\dagger \alpha_{\mathbf{k}_2} \alpha_{\mathbf{k}_1} \end{aligned} \tag{17}$$

where $v(\mathbf{k})$ is the Fourier transform of the interaction potential (which is assumed to be repulsion). Since the interaction is small, new Bose operators can be introduced according to Bogoliubov transformation

$$\begin{aligned} \alpha_{\mathbf{k}} &= \xi_{\mathbf{k}} \cosh \chi_{\mathbf{k}} + \xi_{-\mathbf{k}}^\dagger \sinh \chi_{\mathbf{k}}, \\ \alpha_{-\mathbf{k}}^\dagger &= \xi_{\mathbf{k}} \sinh \chi_{\mathbf{k}} + \xi_{-\mathbf{k}}^\dagger \cosh \chi_{\mathbf{k}}, \end{aligned} \tag{18}$$

with the parameters $\chi_{\mathbf{k}}$ defined as

$$\tanh 2\chi_{\mathbf{k}} = -\frac{v(\mathbf{k}) n_B}{E_{\mathbf{k}} + v(\mathbf{k}) n_B}, \tag{19}$$

where $n_B$ is the average number of particles belonging to Bose-Einstein condensate contained in the volume $V_0$. Provided that the interaction is small (or the gas density $M/V$ is small), almost all particles belong to condensate, so we can take $n_B = MV_0/V$ with relative accuracy of order of $\sqrt{v(0)M/V}$ [9]. By using Bogoliubov transformation, we pass to the ideal gas of new excitations with the dispersion law

$$\omega_{\mathbf{k}} = \sqrt{E_{\mathbf{k}}^2 + 2E_{\mathbf{k}} v(\mathbf{k}) n_B}. \tag{20}$$

Again, we consider dynamical behavior of one-particle density matrix. The way of calculations remains essentially the same as for ideal Bose-gas. In so doing, we obtain the result:

$$\begin{aligned} \rho_n(\mathbf{r}, \mathbf{r}', t) = \frac{n_B}{(n!)^2} \frac{\partial^{2n}}{\partial z^n \partial z'^n} \Big\{ &[1 + (z-1) G(\mathbf{r}, t)] \\ &\times [1 + (z'-1) G^*(\mathbf{r}', t)] \\ &\times \exp[n_B X(z, z')] \Big\}_{z=z'=0} \end{aligned} \tag{21}$$

where we used notations

$$X(z,z') = B(zz'-1) + (1-B)(z+z'-2) \tag{22}$$
$$+A\left[(z-1)^2 + (z'-1)^2\right], \tag{23}$$
$$A = \frac{V_0}{2V}\sum_{\mathbf{k}} \frac{v(\mathbf{k})n_B}{\omega_{\mathbf{k}}},$$
$$B = \frac{V_0}{2V}\sum_{\mathbf{k}} \left[1 + \frac{E_{\mathbf{k}} + v(\mathbf{k})n_B}{\omega_{\mathbf{k}}}\right],$$

and $G(\mathbf{r},t)$ is the Green's function of weakly interacting Bose-gas:

$$G(\mathbf{r},t) = \sum_{\mathbf{k}} \exp(i\mathbf{kr}) \times \left\{\cos\omega_{\mathbf{k}}t - i\frac{E_{\mathbf{k}} + v(\mathbf{k})n_B}{\omega_{\mathbf{k}}}\sin\omega_{\mathbf{k}}t\right\}. \tag{24}$$

Note that for ideal Bose-gas $A = 0$ and $B = 1$, so that the result (10) can be reproduced.

Again, we see that the decoherence wave propagating in the system follows the dynamics of the Green's function (24). Dynamic behavior of $G(\mathbf{r},t)$ at large times $t$ and large distances $r$ can be analyzed by the method of stationary phase [10]. According to this method, the value of the function $G(r,t)$ at the point $\mathbf{r}$ at the instant $t$ is determined by those excitations which have a group velocity $\mathbf{u}(\mathbf{k}) \equiv d\omega_{\mathbf{k}}/d\mathbf{k}$ obeying the requirement

$$\mathbf{u}(\mathbf{k}) = \mathbf{r}/t. \tag{25}$$

The excitations with large wavevectors $\mathbf{k}$ are subject to considerable damping [9], so that at large distances only the undamped long-wavelength excitations determine the dynamics of the Green's function. These excitations represent sound propagating in Bose-gas with the velocity $c = \sqrt{n_B v(0)}/m$. Therefore, we conclude that the decoherence wave in a system of weakly interacting bosons propagates with the sound velocity $c$.

This result can be interpreted in the same way as the appearance of the diffusion decoherence wave in ideal Bose-gas. The measurement affects the particles situated at $\mathbf{r} =$. Due to the interparticle interaction, the decohering perturbation exerted on these particles is transferred to other regions of the system. This transfer of decoherence is governed mainly by the undamped excitations present in the system, i.e. by the long-wavelength excitations propagating with the sound velocity $c$.

Summarizing, we study the decohering influence of local measurement performed on a distributed system situated initially in a pure quantum state. We show that the decohering perturbation exerted on the measured region can propagate over the system by forming the *decoherence wave*. Propagation of the decoherence wave is a dynamic process refering to the local properties at different points of the system, it proceeds with finite velocity and is caused by real physical interactions. It is fundamentally different from such processes as wave function collapse, which characterize the change of the system as a whole, occur right at the instant of measurement, and propagate over the system with "infinite" velocity. Considering Bose-Einstein condensate as an example of the distributed quantum system, we find that the decoherence propagation is governed by the dynamical behavior of the Green's function of the system. Therefore, in ideal Bose-gas decoherence propagates via diffusion, while in the gas of weakly interacting bosons the decoherence wave travels the sound velocity. The results obtained can be checked in real experiments using, e.g. the Bose-Einstein condensate of trapped ions cooled down to very low temperature.

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